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# EXPLICIT FORMS OF SOME FUNCTIONS ARISING IN THE ANALYSIS OF RESONANT SATELLITE ORBITS

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## SUMMARY

The analysis of resonant satellite orbits has been pursued for 18 years, and has led to the most accurate values available for lumped geopotential harmonics of the relevant orders. The basic theory for the resonance effects was developed in the 1960s, but the detailed application of the technique calls for a systematic notation and for the evaluation of two subsidiary functions, namely $\bar{F}$, a function of the orbital inclination, and $G$, a function of the eccentricity. The present paper sets out explicitly the variations in inclination and eccentricity produced by relevant harmonics at the most common resonances (15:1, 14:1, 16:1, 29:2 and 31:2), using the notation that has become standardized in recent years. The paper also gives appropriate expressions for calculating $\bar{F}$ and $G$, with a new Fortran program GQUAD for evaluating $G$.

Departmental Reference:  Space 671

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1    INTRODUCTION

The orbit of an Earth satellite is in resonance with the gravitational field when the track of the satellite over the Earth repeats after an integral number of revolutions. Daily repetitions give the strongest effects, but 2-day repetitions are also of interest. A theory for such resonant orbits was developed in the late 1960s by Allan[1-3] and has been much used in analysing orbits that experience resonance as they decay under the action of air drag. The analyses have yielded values of lumped geopotential harmonics of relevant order, and these values are usually of better accuracy than can currently be achieved by any other method. The first such analysis, by Gooding[4] in 1971, treated the 15th-order resonance of Ariel 3, and the orbital resonances of about 35 other satellites have subsequently been analysed, to determine lumped harmonics, chiefly those of order 14, 15, 16, 29, 30 and 31: from these analyses individual harmonic coefficients have been evaluated for orders 14, 15, 16 and 30 - see Refs 5 to 7 and the papers referred to therein. Nearly always it is the orbital inclination i and eccentricity e that have been analysed.

Allan's theory is in a generalized format, and when analysing specific resonances it is necessary to have a systematic notation and to decide how to evaluate the functions F and G that arise. The present paper gives explicit forms for the rates of change of inclination and eccentricity at the resonances most frequently analysed. (The expressions are largely from a list that has been used in manuscript since 1974.) The evaluation of F is discussed and a useful recurrence relation is given. A Fortran program for the evaluation of G is listed, together with series expansions suitable for small enough e .

2    STANDARD NOTATION

A satellite orbit is said to experience $\beta{:}\alpha$ resonance when, loosely speaking, the ground track repeats after $\beta$ revolutions and $\alpha$ days. Thus 15:1 resonance, also known as 15th-order resonance, implies that the track repeats daily, after 15 revolutions. Similarly, 29:2 resonance occurs when the track repeats every 2 days, after 29 revolutions. Diagrams showing the orbital periods and semi-major axes for exact resonance have been given by Allan[3].

The longitude-dependent part of the geopotential at an exterior point $(r,\theta,\lambda)$ can be written as[8]

$$\frac{\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} \left(\frac{R}{r}\right)^{\ell} P_{\ell}^{m} (\cos\theta)\, N_{\ell m}\left\{\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda\right\} \quad , \tag{1}$$

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where $r$ is the distance from the Earth's centre, $\theta$ is co-latitude, $\lambda$ is longitude (positive to the east), $\mu$ is the gravitational constant for the Earth (398600 $km^3/s^2$), $R$ is the Earth's equatorial radius (6378.1 km), $P_\ell^m(\cos\theta)$ is the associated Legendre function of order $m$ and degree $\ell$, and $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ are the normalized tesseral harmonic coefficients. The normalizing factor $N_{\ell m}$ is given by

$$N_{\ell m}^2 = \frac{2(2\ell + 1)(\ell - m)!}{(\ell + m)!} , \qquad (2)$$

when $m > 0$.

The variations in the orbital elements for near-resonant orbits depend primarily on the resonance angle $\Phi$ defined by

$$\Phi = \alpha(\omega + M) + \beta(\Omega - \nu) , \qquad (3)$$

where $\omega$ is the argument of perigee, $M$ the mean anomaly, $\Omega$ the right ascension of the ascending node and $\nu$ the sidereal angle. Exact resonance occurs when $\dot{\Phi} = 0$, and in practice the effects of resonance are usually significant when $\dot{\Phi}$ is between $-10$ and $+10$ deg/day.

The general term $U_{\ell m}$, say, in equation (1) may be written in the form

$$U_{\ell m} = \sum_{p=0}^{\ell} \sum_{q=-\infty}^{\infty} \frac{\mu}{a}\left(\frac{R}{a}\right)^\ell \bar{F}_{\ell m p} G_{\ell p q} \mathcal{R}\left[j^{\ell-m}(\bar{C}_{\ell m} - j\bar{S}_{\ell m}) \exp\{j(\gamma\Phi - q\omega)\}\right] , \qquad (4)$$

where $a$ is the semi major axis, $\mathcal{R}$ denotes 'real part of', $j = \sqrt{-1}$ and the quantities $\bar{F}$, $G$, $\gamma$, $p$ and $q$ will be discussed later. With this notation, it can be shown that the rate of change of inclination $i$ caused by each pertinent pair of coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$, near $\beta{:}\alpha$ resonance, is given by[3]

$$\frac{di}{dt} = \frac{n(1 - e^2)^{-\frac{1}{2}}}{\sin i}\left(\frac{R}{a}\right)^\ell \bar{F}_{\ell m p} G_{\ell p q}(k \cos i - m)\mathcal{R}\left[j^{\ell-m+1}(\bar{C}_{\ell m} - j\bar{S}_{\ell m}) \exp\{j(\gamma\Phi - q\omega)\}\right] ,$$

$$\dots\dots(5)$$

where $n = \dot{M}$, and $e$ is the eccentricity of the orbit.

The indices $\gamma$, $k$, $p$ and $q$ in equation (5) are integers, with $\gamma$ taking the values 1, 2, 3,... and $q$ the values 0, $\pm 1$, $\pm 2$,... . The equations linking $\ell$, $m$, $k$ and $p$ are:

$$m = \gamma\beta$$

$$k = \gamma\alpha - q$$

$$\ell = k + 2p$$

$$\text{(6)}$$

For a specific resonance, with $\alpha$ and $\beta$ known, the choice of $\gamma$ defines the relevant value of $m$ (and $\gamma = 1$ is nearly always dominant); the choice of a particular $q$ then defines $k$ (with $q = 0$ or $q = \pm 1$ nearly always dominant). For each chosen pair of values of $(\gamma, q)$ there is a range of legitimate values for $\ell$, defined by the two requirements that $\ell \geqslant m$, from equation (1), and that $\ell - k$ is even, from the last of equations (6). (Also $\ell \geqslant k$, but this is nearly always weaker than $\ell \geqslant m$.) Thus (assuming $k < m$), the lowest possible value of $\ell$, $\ell_0$ say, is either $m$ or $m + 1$, and $\ell_0$, $\ell_0 + 2$, $\ell_0 + 4$, ... then all contribute to $di/dt$. As $\ell - k$ must be even,

$$\ell_0 = m \qquad \text{if} \quad m - k \text{ is even}$$

$$\ell_0 = m + 1 \quad \text{if} \quad m - k \text{ is odd}$$

$$\text{(7)}$$

For near-polar orbits the $\ell = \ell_0$ term in $di/dt$ is usually the largest; as the inclination decreases, however, higher-degree terms tend to dominate.

We now turn to the two symbols in equation (4) that have not yet been defined. The first, $\bar{F}_{\ell m p}$, is the normalized inclination function given by[3]

$$\bar{F}_{\ell m p} = \frac{N_{\ell m}(\ell + m)!}{2^\ell p!(\ell - p)!} \sum_\sigma (-1)^\sigma \binom{\ell + k}{\sigma}\binom{\ell - k}{\ell - m - \sigma} (\cos \tfrac{1}{2} i)^{2\ell - m + k - 2\sigma}(\sin \tfrac{1}{2} i)^{m - k + 2\sigma} \,,$$

$$\ldots\ldots\text{(8)}$$

where $\binom{n}{r}$ denotes the usual binomial coefficient $\dfrac{n!}{(n - r)!r!}$ and the summation is over all values of $\sigma$ from $\max(0, k-m)$ to $\min(\ell-m, \ell+k)$. (Note that if $m$ is large, $> 10$ say, it is most unlikely that terms with $|k| > m$ will arise: thus the summation is nearly always from $0$ to $\ell - m$.) When $\ell = m$, equation (8) takes the simple form

$$\bar{F}_{mmp} = \frac{\{2(2m+1)!\}^{\frac{1}{2}}}{2^m p!(m - p)!} (\cos \tfrac{1}{2} i)^{m+k}(\sin \tfrac{1}{2} i)^{m-k} \,; \qquad \text{(9)}$$

but in general there are many terms in the series, and $\bar{F}$ is best evaluated bv means of a recurrence relation[9], as explained in section 6.

The second of the symbols $G_{\ell pq}$ , is a function of eccentricity defined by Kaula[8], and is the same as the Hansen coefficient $X_{k+q}^{-\ell-1,k}$ to be found in the textbooks of celestial mechanics (eg Ref 10). As $G_{\ell pq}$ is of order $e^{|q|}$ and e is usually small in the resonances analysed, values of q greater than 2 are rarely significant in practice. Section 7 describes methods for the accurate and approximate evaluation of $G_{\ell pq}$ .

The same functions $\bar{F}$ and $G$ arise in the equation for the rate of change of eccentricity due to a relevant pair of coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ , near $\beta{:}\alpha$ resonance. From equation (58) of Ref 3, we have:

$$\frac{de}{dt} = ne^{-1}(1-e^2)^{\frac{1}{2}}\left(\frac{R}{a}\right)^{\ell}\bar{F}_{\ell mp}G_{\ell pq}\{(k+q)(1-e^2)^{\frac{1}{2}}-k\}\mathcal{R}\left[\cdot j^{\ell-m+1}(\bar{C}_{\ell m}-j\bar{S}_{\ell m}) \times \right.$$
$$\left. \times \exp\{j(\gamma\Phi-q\omega)\}\right] \quad . \quad (10)$$

## 3   TWO NEW SYMBOLS

To save space, we introduce new symbols for two quantities that are required throughout sections 4 and 5.

First, we write

$$B_{\ell} = n(1-e^2)^{-\frac{1}{2}}\left(\frac{R}{a}\right)^{\ell} , \quad (11)$$

to abbreviate the multiplying factor that applies to both di/dt and de/dt .

Second, we write

$$H = \bar{C}_{\ell m} - j\bar{S}_{\ell m} , \quad (12)$$

the symbol H being intended to signify a (combined) harmonic coefficient. Logically, H should have the suffix $\ell m$ , but this is dropped because $\ell$ and m are the same as in $\bar{F}_{\ell mp}$ and these are always given explicitly. With this notation, the real part of the quantity in square brackets in equation (5) may be expressed, for $\ell = m$ , as

$$\mathcal{R}\left[jH \exp j(\gamma\Phi-q\omega)\right] = -\{\bar{C}_{\ell m} \sin(\gamma\Phi-q\omega) - \bar{S}_{\ell m} \cos(\gamma\Phi-q\omega)\} \quad (13)$$

and, for $\ell = m + 1$ , as

$$\mathcal{R}\left[j^2 H \exp j(\gamma\Phi - q\omega)\right] = -\{\bar{S}_{\ell m} \sin(\gamma\Phi - q\omega) + \bar{C}_{\ell m} \cos(\gamma\Phi - q\omega)\} . \quad (14)$$

## 4   EXPLICIT FORMS FOR THE TERMS IN di/dt

For a specified resonance, the terms with $(\gamma,q) = (1,0)$ are usually the most important in equation (5), followed by the terms with $(\gamma,q) = (2,0)$, $(1,-1)$ and $(1,1)$. All these terms, and some others, are given in the lists below.

For each $(\gamma,q)$ there are contributions from pairs of harmonic coefficients $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ of degree $\ell_0$, $\ell_0 + 2$, $\ell_0 + 4$ , ..., and it is necessary to include many of these if the orbit is of low inclination. The lists below give, for each $(\gamma,q)$, only the term with $\ell = \ell_0$ : the procedure for generating the terms with higher $\ell$ is summarized at the end of this section. In the lists, which now follow, we have written $j^2$ explicitly (rather than as $-1$), to indicate how the formulae run.

15:1 resonance              $\Phi = \omega + M + 15(\Omega - \nu)$

$(\gamma,q) = (1,0)$   $B_{15}\bar{F}_{15,15,7}G_{15,7,0}(\cot i - 15 \operatorname{cosec} i)\mathcal{R}[jH \exp j\Phi]$

$(\gamma,q) = (2,0)$   $B_{30}\bar{F}_{30,30,14}G_{30,14,0}(2 \cot i - 30 \operatorname{cosec} i)\mathcal{R}[jH \exp 2j\Phi]$

$(\gamma,q) = (3,0)$   $B_{45}\bar{F}_{45,45,21}G_{45,21,0}(3 \cot i - 45 \operatorname{cosec} i)\mathcal{R}[jH \exp 3j\Phi]$

$(\gamma,q) = (1,-1)$   $B_{16}\bar{F}_{16,15,7}G_{16,7,-1}(2 \cot i - 15 \operatorname{cosec} i)\mathcal{R}[j^2H \exp j(\Phi + \omega)]$

and $(1,1)$   $B_{16}\bar{F}_{16,15,8}G_{16,8,1}(0 - 15 \operatorname{cosec} i)\mathcal{R}[j^2H \exp j(\Phi - \omega)]$

$(\gamma,q) = (2,-1)$   $B_{31}\bar{F}_{31,30,14}G_{31,14,-1}(3 \cot i - 30 \operatorname{cosec} i)\mathcal{R}[j^2H \exp j(2\Phi + \omega)]$

and $(2,1)$   $B_{31}\bar{F}_{31,30,15}G_{31,15,1}(\cot i - 30 \operatorname{cosec} i)\mathcal{R}[j^2H \exp j(2\Phi - \omega)]$

$(\gamma,q) = (1,-2)$   $B_{15}\bar{F}_{15,15,6}G_{15,6,-2}(3 \cot i - 15 \operatorname{cosec} i)\mathcal{R}[jH \exp j(\Phi + 2\omega)]$

and $(1,2)$   $B_{15}\bar{F}_{15,15,8}G_{15,8,2}(- \cot i - 15 \operatorname{cosec} i)\mathcal{R}[jH \exp j(\Phi - 2\omega)]$

$(\gamma,q) = (2,-2)$   $B_{30}\bar{F}_{30,30,13}G_{30,13,-2}(4 \cot i - 30 \operatorname{cosec} i)\mathcal{R}[jH \exp 2j(\Phi + \omega)]$

and $(2,2)$   $B_{30}\bar{F}_{30,30,15}G_{30,15,2}(0 - 30 \operatorname{cosec} i)\mathcal{R}[jH \exp 2j(\Phi - \omega)]$

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**14:1 resonance** $\qquad \Phi = \omega + M + 14(\Omega - \nu)$

$(\gamma,q) = (1,0) \qquad B_{15}\bar{F}_{15,14,7}G_{15,7,0}(\cot i - 14 \csc i)\mathcal{R}[j^2 H \exp j\Phi]$

$(\gamma,q) = (2,0) \qquad B_{28}\bar{F}_{28,28,13}G_{28,13,0}(2 \cot i - 28 \csc i)\mathcal{R}[jH \exp 2j\Phi]$

$(\gamma,q) = (1,-1) \qquad B_{14}\bar{F}_{14,14,6}G_{14,6,-1}(2 \cot i - 14 \csc i)\mathcal{R}[jH \exp j(\Phi + \omega)]$

$\qquad$ and $(1,1) \qquad B_{14}\bar{F}_{14,14,7}G_{14,7,1}(0 - 14 \csc i)\mathcal{R}[jH \exp j(\Phi - \omega)]$

$(\gamma,q) = (2,-1) \qquad B_{29}\bar{F}_{29,28,13}G_{29,13,-1}(3 \cot i - 28 \csc i)\mathcal{R}[j^2 H \exp j(2\Phi + \omega)]$

$\qquad$ and $(2,1) \qquad B_{29}\bar{F}_{29,28,14}G_{29,14,1}(\cot i - 28 \csc i)\mathcal{R}[j^2 H \exp j(2\Phi - \omega)]$

$(\gamma,q) = (1,-2) \qquad B_{15}\bar{F}_{15,14,6}G_{15,6,-2}(3 \cot i - 14 \csc i)\mathcal{R}[j^2 H \exp j(\Phi + 2\omega)]$

$\qquad$ and $(1,2) \qquad B_{15}\bar{F}_{15,14,8}G_{15,8,2}(- \cot i - 14 \csc i)\mathcal{R}[j^2 H \exp j(\Phi - 2\omega)]$


**16:1 resonance** $\qquad \Phi = \omega + M + 16(\Omega - \nu)$

$(\gamma,q) = (1,0) \qquad B_{17}\bar{F}_{17,16,8}G_{17,8,0}(\cot i - 16 \csc i)\mathcal{R}[j^2 H \exp j\Phi]$

$(\gamma,q) = (2,0) \qquad B_{32}\bar{F}_{32,32,15}G_{32,15,0}(2 \cot i - 32 \csc i)\mathcal{R}[jH \exp 2j\Phi]$

$(\gamma,q) = (1,-1) \qquad B_{16}\bar{F}_{16,16,7}G_{16,7,-1}(2 \cot i - 16 \csc i)\mathcal{R}[jH \exp j(\Phi + \omega)]$

$\qquad$ and $(1,1) \qquad B_{16}\bar{F}_{16,16,8}G_{16,8,1}(0 - 16 \csc i)\mathcal{R}[jH \exp j(\Phi - \omega)]$

29:2 resonance $\qquad \Phi = 2(\omega + M) + 29(\Omega - \nu)$

$(\gamma, q) = (1,0) \quad B_{30}\bar{F}_{30,29,14}G_{30,14,0}(2 \cot i - 29 \csc i)\mathcal{R}[j^2H \exp j\Phi]$

$(\gamma, q) = (2,0) \quad B_{58}\bar{F}_{58,58,27}G_{58,27,0}(4 \cot i - 58 \csc i)\mathcal{R}[jH \exp 2j\Phi]$

$(\gamma, q) = (1,-1) \quad B_{29}\bar{F}_{29,29,13}G_{29,13,-1}(3 \cot i - 29 \csc i)\mathcal{R}[jH \exp j(\Phi + \omega)]$

$\qquad$ and $(1,1) \quad B_{29}\bar{F}_{29,29,14}G_{29,14,1}(\cot i - 29 \csc i)\mathcal{R}[jH \exp j(\Phi - \omega)]$

$(\gamma, q) = (2,-1) \quad B_{59}\bar{F}_{59,58,27}G_{59,27,-1}(5 \cot i - 58 \csc i)\mathcal{R}[j^2H \exp j(2\Phi + \omega)]$

$\qquad$ and $(2,1) \quad B_{59}\bar{F}_{59,58,28}G_{59,28,1}(3 \cot i - 58 \csc i)\mathcal{R}[j^2H \exp j(2\Phi - \omega)]$

$(\gamma, q) = (1,-2) \quad B_{30}\bar{F}_{30,29,13}G_{30,13,-2}(4 \cot i - 29 \csc i)\mathcal{R}[j^2H \exp j(\Phi + 2\omega)]$

$\qquad$ and $(1,2) \quad B_{30}\bar{F}_{30,29,15}G_{30,15,2}(0 - 29 \csc i)\mathcal{R}[j^2H \exp j(\Phi - 2\omega)]$

31:2 resonance $\qquad \Phi = 2(\omega + M) + 31(\Omega - \nu)$

$(\nu, q) = (1,0) \quad B_{32}\bar{F}_{32,31,15}G_{32,15,0}(2 \cot i - 31 \csc i)\mathcal{R}[j^2H \exp j\Phi]$

$(\gamma, q) = (2,0) \quad B_{62}\bar{F}_{62,62,29}G_{62,29,0}(4 \cot i - 62 \csc i)\mathcal{R}[jH \exp 2j\Phi]$

$(\gamma, q) = (1,-1) \quad B_{31}\bar{F}_{31,31,14}G_{31,14,-1}(3 \cot i - 31 \csc i)\mathcal{R}[jH \exp j(\Phi + \omega)]$

$\qquad$ and $(1,1) \quad B_{31}\bar{F}_{31,31,15}G_{31,15,1}(\cot i - 31 \csc i)\mathcal{R}[jH \exp j(\Phi - \omega)]$

As stated at the beginning of this section, the lists give only the terms for $\ell = \ell_0$, and there are additional terms for $\ell = \ell_0 + 2$, $\ell_0 + 4$, $\ell_0 + 6$, ... . These additional terms may easily be derived from equation (5) which may be rewritten as

$$\frac{di}{dt} = (k \cot i - m \csc i)B_\ell \bar{F}_{\ell mp}G_{\ell pq}\mathcal{R}[j^{\ell-m+1}H_{\ell m} \exp j(\gamma\Phi - q\omega)] ; \qquad (15)$$

here $B_\ell$ is given by (11), and the suffix $\ell m$ has been restored to $H$, which is given by (12). For specified values of $(\alpha, \beta)$ and $(\gamma, q)$, the indices $k$ and $m$

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in equation (15) are constant, and it is helpful to gather the terms of degree $\ell_0$, $\ell_0 + 2$, $\ell_0 + 4$, ... into a 'lumped harmonic'. We write these lumped harmonics as

$$\bar{C}_m^{q,k} = \sum_\ell Q_\ell^{q,k} \bar{C}_{\ell m} \quad \text{and} \quad \bar{S}_m^{q,k} = \sum_\ell Q_\ell^{q,k} \bar{S}_{\ell m} \,, \qquad (16)$$

where $\ell$ increases in steps of 2 from its minimum permissible value, $\ell_0$ , and it is convenient always to take $Q_{\ell_0}^{q,k} = 1$ . Then we see, from equation (15), that

$$Q_\ell^{q,k} = \frac{B_\ell \bar{F}_{\ell m p} G_{\ell p q}}{B_{\ell_0} \bar{F}_{\ell_0 m p_0} G_{\ell_0 p_0 q}} (-1)^{\frac{1}{2}(\ell - \ell_0)} \,. \qquad (17)$$

The series of terms that arises is best indicated by an example: we choose 14:1 resonance and the term with $(\gamma,q) = (1,-1)$ , for which $k = 2$ from (6). The contribution of this term to $di/dt$ is given by

$$\frac{di}{dt} = 2B_{14}\bar{F}_{14,14,6}G_{14,6,-1}(7\cosec i - \cot i)\left\{\bar{C}_{14}^{-1,2}\sin(\phi + \omega) - \bar{S}_{14}^{-1,2}\cos(\phi + \omega)\right\}, \qquad (18)$$

where equation (13) establishes the terms in curly brackets and

$$\bar{C}_{14}^{-1,2} = \bar{C}_{14,14} - \left(\frac{R}{a}\right)^2 \frac{\bar{F}_{16,14,7}G_{16,7,-1}}{\bar{F}_{14,14,6}G_{14,6,-1}} \bar{C}_{16,14} + \left(\frac{R}{a}\right)^4 \frac{\bar{F}_{18,14,8}G_{18,8,-1}}{\bar{F}_{14,14,6}G_{14,6,-1}} \bar{C}_{18,14} - \cdots$$

$$\cdots \cdots (19)$$

and similarly for $S$ . Explicit forms for other $\bar{C}_m^{q,k}$ may be found in Refs 11 and 12.

## 5   EXPLICIT FORMS FOR THE TERMS IN $de/dt$

If we introduce $B_\ell$ from equation (11) and $H$ from equation (12), the expression (10) for $de/dt$ becomes

$$\frac{de}{dt} = B_\ell \bar{F}_{\ell m p} G_{\ell p q} e^{-1}(1 - e^2)\{(k + q)(1 - e^2)^{\frac{1}{2}} - k\}\Re[j^{\ell-m+1} H \exp j(\gamma\phi - q\omega)] \,. \qquad (20)$$

On expanding in powers of $e$ and replacing $k$ by $\gamma\alpha - q$, we have

$$\frac{de}{dt} = B_\ell \bar{F}_{\ell m p} G_{\ell p q} e^{-1} \{q - \tfrac{1}{2}e^2(\gamma\alpha + 2q) + \tfrac{3}{8}e^4\gamma\alpha + \ldots\}\Re[j^{\ell-m+1}H \exp j(\gamma\Phi - q\omega)] \ . \quad (21)^*$$

In the explicit forms below we ignore terms that are $O(e^2)$ relative to the main term. When $q \neq 0$, the necessary correction factor is $\{1 - \tfrac{1}{2}e^2(2 + \gamma\alpha/q) + O(e^4)\}$. When $q = 0$, the correction factor is $\{1 - \tfrac{3}{8}e^2 + O(e^4)\}$. The expansion in powers of $e^2$ is very helpful because $e < 0.01$ for most of the orbits analysed. For larger values of $e$, the unexpanded form can be used: the term in curly brackets in (21) should then be replaced by $(1 - e^2)\{q + \gamma\alpha(1 - e^2)^{\frac{1}{2}} - \gamma\alpha\}$.

Again the lists give, for each $(\gamma,q)$, only the term with $\ell = \ell_0$.

For a specified resonance, the terms with $(\gamma,q) = (1,-1)$ and $(1,1)$ in equation (21) are usually the most important for a low-eccentricity orbit $(e < 0.2)$. The terms next in importance are usually those with $(\gamma,q) = (2,-1)$ and $(2,1)$; the terms with $(\gamma,q) = (1,-2)$ and $(1,2)$ may also be significant when $e > 0.01$.

### 15:1 resonance $\qquad\qquad \Phi = \omega + M + 15(\Omega - \nu)$

$(\gamma,q) = (1,-1) \qquad -B_{16}\bar{F}_{16,15,7}G_{16,7,-1}e^{-1}\Re[j^2 H \exp j(\Phi + \omega)]$

$\qquad\qquad$ and $(1,1) \qquad B_{16}\bar{F}_{16,15,8}G_{16,8,1}e^{-1}\Re[j^2 H \exp j(\Phi - \omega)]$

$(\gamma,q) = (2,-1) \qquad -B_{31}\bar{F}_{31,30,14}G_{31,14,-1}e^{-1}\Re[j^2 H \exp j(2\Phi + \omega)]$

$\qquad\qquad$ and $(2,1) \qquad B_{31}\bar{F}_{31,3C,15}G_{31,15,1}e^{-1}\Re[j^2 H \exp j(2\Phi - \omega)]$

$(\gamma,q) = (3,-1) \qquad -B_{46}\bar{F}_{46,45,21}G_{46,21,-1}e^{-1}\Re[j^2 H \exp j(3\Phi + \omega)]$

$(\gamma,q)$ and $(3,1) \qquad B_{46}\bar{F}_{46,45,22}G_{46,22,1}e^{-1}\Re[j^2 H \exp j(3\Phi - \omega)]$

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* Note that the coefficient of $\tfrac{1}{2}e^2$, namely $(\gamma\alpha + 2q)$, is equal to $(k + 3q)$, whereas $(k + q)$ has been given (incorrectly) in several previous publications, beginning with Ref 13. The error is of no consequence, however, because the term is used only when $q = 0$, and then both the correct and the incorrect forms reduce to $k$.

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$(\gamma,q) = (1,-2)$ $\quad -2B_{15}\bar{F}_{15,15,6}G_{15,6,-2}e^{-1}\mathcal{R}[jH \exp j(\phi + 2\omega)]$

$\quad$ and $(1,2)$ $\quad 2B_{15}\bar{F}_{15,15,8}G_{15,8,2}e^{-1}\mathcal{R}[jH \exp j(\phi - 2\omega)]$

$(\gamma,q) = (1,0)$ $\quad -\frac{1}{2}B_{15}\bar{F}_{15,15,7}G_{15,7,0}e\mathcal{R}[jH \exp j\phi]$

$(\gamma,q) = (2,0)$ $\quad -B_{30}\bar{F}_{30,30,14}G_{30,14,0}e\mathcal{R}[jH \exp 2j\phi]$

$(\gamma,q) = (2,-2)$ $\quad -2B_{30}\bar{F}_{30,30,13}G_{30,13,-2}e^{-1}\mathcal{R}[jH \exp 2j(\phi + \omega)]$

$\quad$ and $(2,2)$ $\quad 2B_{30}\bar{F}_{30,30,15}G_{30,15,2}e^{-1}\mathcal{R}[jH \exp 2j(\phi - \omega)]$

**14:1 resonance** $\qquad\qquad \phi = \omega + M + 14(\Omega - \nu)$

$(\gamma,q) = (1,-1)$ $\quad -B_{14}\bar{F}_{14,14,6}G_{14,6,-1}e^{-1}\mathcal{R}[jH \exp j(\phi + \omega)]$

$\quad$ and $(1,1)$ $\quad B_{14}\bar{F}_{14,14,7}G_{14,7,1}e^{-1}\mathcal{R}[jH \exp j(\phi - \omega)]$

$(\gamma,q) = (2,-1)$ $\quad -B_{29}\bar{F}_{29,28,13}G_{29,13,-1}e^{-1}\mathcal{R}[j^2H \exp j(2\phi + \omega)]$

$\quad$ and $(2,1)$ $\quad B_{29}\bar{F}_{29,28,14}G_{29,14,1}e^{-1}\mathcal{R}[j^2H \exp j(2\phi - \omega)]$

**16:1 resonance** $\qquad\qquad \phi = \omega + M + 16(\Omega - \nu)$

$(\gamma,q) = (1,-1)$ $\quad -B_{16}\bar{F}_{16,16,7}G_{16,7,-1}e^{-1}\mathcal{R}[jH \exp j(\phi + \omega)]$

$\quad$ and $(1,1)$ $\quad B_{16}\bar{F}_{16,16,8}G_{16,8,1}e^{-1}\mathcal{R}[jH \exp j(\phi - \omega)]$

**29:2 resonance** $\qquad\qquad \phi = 2(\omega + M) + 29(\Omega - \nu)$

$(\gamma,q) = (1,-1)$ $\quad -B_{29}\bar{F}_{29,29,13}G_{29,13,-1}e^{-1}\mathcal{R}[jH \exp j(\phi + \omega)]$

$\quad$ and $(1,1)$ $\quad B_{29}\bar{F}_{29,29,14}G_{29,14,1}e^{-1}\mathcal{R}[jH \exp j(\phi - \omega)]$

**31:2 resonance** $\qquad\qquad \phi = 2(\omega + M) + 31(\Omega - \nu)$

$(\gamma,q) = (1,-1)$ $\quad -B_{31}\bar{F}_{31,31,14}G_{31,14,-1}e^{-1}\mathcal{R}[jH \exp j(\phi + \omega)]$

$\quad$ and $(1,1)$ $\quad B_{31}\bar{F}_{31,31,15}G_{31,15,1}e^{-1}\mathcal{R}[jH \exp j(\phi - \omega)]$

As with  di/dt , the terms above are those for  $\ell = \ell_0$ ; the additional terms, for  $\ell = \ell_0 + 2, \ell_0 + 4, \ell_0 + 6, \ldots$, can best be expressed in terms of the lumped harmonics defined in equations (16) and (17).  Again it may be useful to give an example: we choose 15:1 resonance and the terms with $(\gamma,q) = (2,1)$, for which  $k = 1$  from equation (6).  The total contribution of this term to  de/dt is given by

$$\frac{de}{dt} = -B_{31}\bar{F}_{31,30,15}G_{31,15,1}e^{-1}\left\{\bar{S}_{30}^{1,1} \sin(2\Phi - \omega) + \bar{C}_{30}^{1,1} \cos(2\Phi - \omega)\right\} , \quad (22)$$

where equation (14) establishes the terms in curly brackets.  In equation (22),

$$\bar{S}_{30}^{1,1} = \bar{S}_{31,30} - \left(\frac{R}{a}\right)^2 \frac{\bar{F}_{33,30,16}G_{33,16,1}}{\bar{F}_{31,30,15}G_{31,15,1}} \bar{S}_{33,30} + \left(\frac{R}{a}\right)^4 \frac{\bar{F}_{35,30,17}G_{35,17,1}}{\bar{F}_{31,30,15}G_{31,15,1}} \bar{S}_{35,30} -$$

$$\ldots\ldots(23)$$

and similarly for  C .

## 6  EXPLICIT FORMS FOR THE INCLINATION FUNCTIONS

The inclination function  $\bar{F}_{\ell m p}$  is given by equation (8), but it is convenient to split this into two factors, the first being a series which is free of large numerical values like $(\ell + m)!$ , while the second is a single quantity that provides an efficient combination of the various large values.

The first factor, introduced from Ref 9, is  $A_{\ell m}^k$  (a function of  i ) which, for  $k \leq m$ , is given by equation (14) of Ref 9 as

$$A_{\ell m}^k = \frac{2^m(\ell + m)!(k + m)!}{(2m)!(\ell + k)!S^{m-k}(1 + C)^k} \sum_{\sigma=0}^{\ell-m} (-1)^\sigma \binom{\ell + k}{\sigma}\binom{\ell - k}{\ell - m - \sigma} (\cos \tfrac{1}{2}i)^{2\ell-m+k-2\sigma} \times$$

$$\times (\sin \tfrac{1}{2}i)^{m-k+2\sigma} ,$$

$$\ldots\ldots(24)$$

where  S = sin i  and  C = cos i .  It is shown in Ref 9 that  $A_{\ell m}^k$  satisfies the recurrence relation

$$(\ell - 1)(\ell - m)(\ell + k)A_{\ell m}^k = (2\ell - 1)\{\ell(\ell - 1)C - mk\}A_{\ell-1,m}^k$$

$$- \ell(\ell + m - 1)(\ell - k - 1)A_{\ell-2,m}^k , \quad (25)$$

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for which (when $k \leqslant m$) the starting values are

$$A_{mm}^{k} = 1 \quad , \qquad A_{m+1,m}^{k} = \frac{(2m + 1)\{(m + 1)C - k\}}{m + 1 + k} \quad . \tag{26}$$

Thus $A_{m+2,m}^{k}$ can be found by substituting (26) into (25), with $\ell = m + 2$ ; and so on up the line. (Ref 9 also gives starting values for $m \leqslant k$ , but these are irrelevant here.)

We may now write

$$\bar{F}_{\ell m p} = A_{\ell m}^{k} V_{\ell m}^{k} \quad , \tag{27}$$

where $V_{\ell m}^{k}$ has a 'normalizing' role and, from (8) and (24), is given by

$$V_{\ell m}^{k} = \frac{(2m)!(\ell + k)!S^{m-k}(1 + C)^{k}}{2^{\ell+m}(k + m)! \, [\frac{1}{2}(\ell + k)]! \, [\frac{1}{2}(\ell - k)]!} \left\{ \frac{2(2\ell + 1)(\ell - m)!}{(\ell + m)!} \right\}^{\frac{1}{2}} \quad . \tag{28}$$

The general form of the $\bar{F}$ functions is usefully clarified by the split between V and A . From (26), $A_{mm}^{k}$ is constant; $A_{m+1,m}^{k}$ is linear in $C (= \cos i)$ and has one zero at $C = k/(m+1)$ , which is quite near $i = 90^{\circ}$ if $k \leqslant m$ , as is usual; from (25), $A_{m+2,m}^{k}$ is a quadratic in $C$ , and generally has two zeros; $A_{m+3,m}^{k}$ is a cubic in $C$ , and usually has three zeros; and so on. Thus the variation of $A_{\ell m}^{k}$ with inclination becomes increasingly oscillatory as $\ell$ increases. This oscillatory function is to be multiplied by $V_{\ell m}^{k}$ , in which the term $S^{m-k} = (\sin i)^{m-k}$ dominates the variation with inclination if $k \leqslant m$ . When $m$ is large, there is a strong maximum of $S^{m-k}$ at $i = 90^{\circ}$ , with a rapid decrease at lower inclinations.

This behaviour is illustrated by Figs 1 and 2 (taken from Ref 3) which show the variations of some of the $\bar{F}$ with inclination, for 15th- and 16th-order resonance. The dotted lines at $i = 90^{\circ}$ have been added to make it clear that the curves are not symmetrical about $i = 90^{\circ}$ . More extensive diagrams for 15th order, up to $\ell = 33$ , have been given by Klokočnik[14].

When $\ell \gg m$ , there can be numerical problems associated with the computation of $\bar{F}$ . Methods for avoiding these problems are reviewed in Ref 13.

From the recurrence relations (25) and (26), the values of the $A_{\ell m}^{k}$ required for the $\bar{F}_{\ell m p}$ that occur in the lists of sections 4 and 5 are as follows.

$$A_{15,15}^1 = 1 \qquad A_{30,30}^2 = 1$$

$$A_{16,15}^2 = \frac{31(8C-1)}{9} \qquad A_{16,15}^0 = 31C$$

$$A_{31,30}^3 = \frac{61(31C-3)}{34} \qquad A_{31,30}^1 = \frac{61(31C-1)}{32}$$

$$A_{15,15}^3 = 1 \qquad A_{15,15}^{-1} = 1$$

$$A_{30,30}^4 = 1 \qquad A_{30,30}^0 = 1$$

$$A_{15,14}^1 = \frac{29(15C-1)}{16} \qquad A_{28,28}^2 = 1$$

$$A_{14,14}^2 = 1 \qquad A_{14,14}^0 = 1$$

$$A_{29,28}^3 = \frac{57(29C-3)}{32} \qquad A_{29,28}^1 = \frac{19(29C-1)}{10}$$

$$A_{15,14}^3 = \frac{29(5C-1)}{6} \qquad A_{15,14}^{-1} = \frac{29(15C+1)}{14}$$

$$A_{17,16}^1 = \frac{33(17C-1)}{18} \qquad A_{32,32}^2 = 1$$

$$A_{16,16}^2 = 1 \qquad A_{16,16}^0 = 1$$

$$A_{30,29}^2 = \frac{59(15C-1)}{16} \qquad A_{58,58}^4 = 1$$

$$A_{29,29}^3 = 1 \qquad A_{29,29}^1 = 1$$

$$A_{59,58}^5 = \frac{117(59C-5)}{64} \qquad A_{59,58}^3 = \frac{117(59C-3)}{62}$$

$$A_{30,29}^4 = \frac{59(30C-4)}{34} \qquad A_{30,29}^0 = 59C$$

$$A_{32,31}^2 = \frac{63(32C-2)}{34} \qquad A_{62,62}^4 = 1$$

$$A_{31,31}^3 = 1 \qquad A_{31,31}^1 = 1$$

## 7 EVALUATION OF $G_{\ell pq}$

### 7.1 Accurate computation by quadrature

As explained in the Appendix, the best general expression for the eccentricity function $G_{\ell pq}$ is in terms of a definite integral, as

$$G_{\ell pq} = X_{k+q}^{-\ell-1,k} = \frac{1}{\pi} \int_0^\pi \left(\frac{a}{r}\right)^{\ell+1} \exp j\{kv - (k + q)M\} \, dM \ , \qquad (29)$$

where $v$ is the true anomaly and

$$\frac{a}{r} = \frac{1 + e \cos v}{1 - e^2} \ . \qquad (30)$$

A Fortran program, GQUAD , to evaluate $G_{\ell pq}$ from (29) has been written by A.W. Odell, and a listing is included in the Appendix.

### 7.2 A truncated-series approximation, for small $e$

If $e$ is small, it is possible to express $G_{\ell pq}$ as a power series in $e$ , the main term being of order $e^{|q|}$ with smaller terms of order $e^{|q|+2}$ , $e^{|q|+4}$ , ... . The expressions that arise are very involved, however, and it is not easy to know how many terms are needed in a given application. What *is* fairly easy is to derive a formula for the coefficient of $e^{|q|}$ in the leading term (monomial) of $G_{\ell pq}$ : this monomial is itself expressible as a polynomial in the constant $\gamma\alpha$ . For a given application, the values of $G_{\ell pq}$ thus obtained can be checked against those calculated from the integral (29) to assess the range of validity of the truncated-series approximation.

From equations (59a) and (59b) of Ref 2, the required approximation for $G_{\ell pq}$ , written in terms of $\ell$, $k$ and $q$ , is

$$\left. \begin{aligned} G_{\ell pq} &= (-\tfrac{1}{2}e)^q \sum_{\sigma=0}^{q} \frac{(-k - q)^\sigma}{\sigma!} \binom{-\ell - k}{q - \sigma} + 0(e^{q+2}) \qquad \text{if } q \geqslant 0 \\[2em] &= (-\tfrac{1}{2}e)^{-q} \sum_{\sigma=0}^{-q} \frac{(k + q)^\sigma}{\sigma!} \binom{-\ell + k}{-q - \sigma} + 0(e^{-q+2}) \qquad \text{if } q \leqslant 0 \end{aligned} \right\} \ . \qquad (31)$$

Note that $k + q = \gamma\alpha$ from equation (6). We now ignore the O-terms and write G as $\hat{G}$, as a reminder of this omission. In most applications $|q| \leqslant 2$, and equations (31) then give:

$$\hat{G}_{\ell,p,0} = 1 , \tag{32}$$

$$\left.\begin{aligned}\hat{G}_{\ell,p,-1} &= \tfrac{1}{2}e(\ell - 2k + 1)\\[2mm]\hat{G}_{\ell,p,1} &= \tfrac{1}{2}e(\ell + 2k + 1)\end{aligned}\right\} , \tag{33}$$

$$\left.\begin{aligned}\hat{G}_{\ell,p,-2} &= \tfrac{1}{8}e^2\{(\ell + 1)(\ell + 4) - k(4\ell + 9) + 4k^2\}\\[2mm]\hat{G}_{\ell,p,2} &= \tfrac{1}{8}e^2\{(\ell + 1)(\ell + 4) + k(4\ell + 9) + 4k^2\}\end{aligned}\right\} . \tag{34}$$

For orbits of small eccentricity, all the G functions appearing in the lists in sections 4 and 5 can be evaluated approximately by the use of equations (32) to (34). A selection is given below, with the values of $\gamma\alpha$.

| | | |
|---|---|---|
| $\hat{G}_{16,7,-1} = 6\tfrac{1}{2}e$ | $\hat{G}_{16,8,1} = 8\tfrac{1}{2}e$ | $(\gamma\alpha = 1)$ |
| $\hat{G}_{31,14,-1} = 13e$ | $\hat{G}_{31,15,1} = 17e$ | $(\gamma\alpha = 2)$ |
| $\hat{G}_{15,6,-2} = 16\tfrac{5}{8}e^2$ | $\hat{G}_{15,8,2} = 29\tfrac{7}{8}e^2$ | $(\gamma\alpha = 1)$ |
| $\hat{G}_{30,13,-2} = 75\tfrac{1}{4}e^2$ | $\hat{G}_{30,15,2} = 131\tfrac{3}{4}e^2$ | $(\gamma\alpha = 2)$ |
| $\hat{G}_{14,6,-1} = 5\tfrac{1}{2}e$ | $\hat{G}_{14,7,1} = 7\tfrac{1}{2}e$ | $(\gamma\alpha = 1)$ |
| $\hat{G}_{29,13,-1} = 12e$ | $\hat{G}_{29,14,1} = 16e$ | $(\gamma\alpha = 2)$ |
| $\hat{G}_{31,14,-1} = 14e$ | $\hat{G}_{31,15,1} = 16e$ | $(\gamma\alpha = 2)$ |

Equation (33) shows that $\hat{G}_{\ell pq}$ is of order $\tfrac{1}{2}\ell e$ for $|q| = 1$, and equation (34) shows that $\hat{G}_{\ell pq}$ is of order $\tfrac{1}{8}\ell^2 e^2$ for $|q| = 2$, assuming $|k| \leqslant \ell$. These numerical values are crucial in assessing the likely importance of terms in $q = \pm 1$ and $q = \pm 2$. For example, if $\ell e > 2$, these 'subsidiary resonances' can be more important than the 'main resonance', as in Wagner's analysis[16] of Vanguard 3, for which $\ell = 11$ and $e = 0.19$.

To generate $\hat{G}_{\ell pq}$ for high values of $q$, it is best to use a recurrence relation. For $q > 0$, it can be shown that

$$\hat{G}_{\ell pq} = \frac{e}{4q} \{e(\ell + 1 + k)\hat{G}_{\ell,p,q-2} + 2(\ell + 1 + 2k)\hat{G}_{\ell,p,q-1}\} \quad , \qquad (35)$$

and, for $q < 0$, that

$$\hat{G}_{\ell pq} = -\frac{e}{4q} \{e(\ell + 1 - k)\hat{G}_{\ell,p,q+2} + 2(\ell + 1 - 2k)\hat{G}_{\ell,p,q+1}\} \quad . \qquad (36)$$

The utility of $\hat{G}$ as an approximation to $G$ is indicated in Figs 3 and 4, in which $G_{\ell pq}/\hat{G}_{\ell pq}$ is plotted against $e$ for selected values of $(\ell,p,q)$. Fig 3 shows that $\hat{G}$ is useless as an approximation for large $e$. But for nearly all the orbits analysed at resonance, $e$ has been less than 0.01, and, as Fig 4 shows, the use of $\hat{G}$ as an approximation for $G$ is rarely in error by more than 1%. As the observational errors in the values derived for the lumped harmonics are 3% or more, the use of $\hat{G}$ does not significantly degrade the accuracy of the analyses.

## 7.3  An accurate recurrence relation for the $G_{\ell pq}$

As already indicated, the analytical derivations of the Hansen functions are very involved, and these functions have been much studied (*eg* Refs 17 to 22). It is largely because of this complexity that the power-series expansions are only satisfactory for small $e$. A number of exact relationships between the functions have been discovered, however, that are free of quadrature. One such relation, given by Giacaglia[20], allows values of $G_{\ell pq}$ to be computed by recurrence from a basic set of values. The relation is

$$G_{\ell pq} = \frac{1}{2k(1 - e^2)^{\frac{1}{2}}} \left\{ 2(k + q)G_{\ell-2,p-1,q} - \frac{(\ell - 1)e}{(1 - e^2)^{\frac{1}{2}}} \left(G_{\ell-1,p-1,q-1} - G_{\ell-1,p,q+1}\right) \right\} \quad .$$

$$(37)*$$

The patterns of the suffixes is not immediately obvious in equation (37); the key is that $k + q$ has the same value for each of the four $G$ functions. Since $k + q = \gamma\alpha$ in resonance analysis, the relation is directly applicable; however, the 'basic values' still have to be computed, and this limits the usefulness of the relation.

---

\* In Ref 20, $(\ell - 1)e$ is wrongly given as $(\ell + 1)e$. We believe that the correct formula was given earlier by P.J. Cefola.

The $G$ functions have simple closed-form expressions when $k + q = 0$. We can distinguish such $G$ functions by suppressing the suffix $p$, so that (for example) $G_{2,0} = (1 - e^2)^{-3/2}$ and $G_{3,1} = G_{3,-1} = e(1 - e^2)^{-5/2}$, whilst it follows from equation (29) that $G_{\ell,\pm\ell} = 0$ if $\ell > 0$. Equation (37) now reduces to

$$G_{\ell q} = \frac{(\ell - 1)e}{2q(1 - e^2)} (G_{\ell-1,q-1} - G_{\ell-1,q+1}), \qquad (38)$$

valid for $q \neq 0$. Taking $\ell = 3$ and $q = \pm 1$ confirms the expressions for $G_{3,1}$ and $G_{3,-1}$ already quoted, given $G_{2,0}$, and leads on to $G_{4,2} = G_{4,-2} = \tfrac{3}{2}e^2(1 - e^2)^{-7/2}$ etc.

## 8    COMMENTS ON THE NOTATION

Any attentive reader of this Report will have noticed a certain ambivalence of notation over the indices $p$ and $k$, one of which is always redundant, because $k + 2p = \ell$. We regard $k$ as the more useful of the parameters, for two reasons. First, most formulae are simpler and more revealing in terms of $k$. Second, $k$ provides symmetries unattainable with $p$: in equation (4), for example, $\sum\limits_{p=0}^{\ell}$ could be replaced by $\sum\limits_{k=-\ell}^{\ell}$, though with the caveat that the summation is in steps of 2, that is, $-\ell, -\ell + 2, \ldots, \ell - 2, \ell$.

So we considered rewriting $\bar{\bar{F}}_{\ell m p}(i)$ as $\bar{\bar{F}}_{\ell m}^{k}(i)$, following Ref 23, and $G_{\ell p q}(e)$ as $G_{\ell q}^{k}(e)$. However, the use of $\bar{\bar{F}}_{\ell m p}$ and $G_{\ell p q}$ has become so widespread that it is now 'standard': this consideration, and the absence of a recognized notation for summation in steps of 2, deterred us from amending the notation, though we have used the affix $k$ with $A$ and $V$ in section 6.

There would also be some advantage in defining an extra symbol to represent $k + q$, which arises in equations (20), (29) and (31). Here we have been able to identify $k + q$ with $\gamma\alpha$; but equations (29) and (31) are independent of resonance and might benefit from the extra symbol.

Finally, we should draw attention to the fact that our definition of, and notation for, the lumped harmonics $\bar{C}_{m}^{q,k}$ and $\bar{S}_{m}^{q,k}$ in equations (16) differs from that adopted by Klokočnik in his extensive studies of resonance (see, for example, Ref 24).

## 9    CONCLUDING SUMMARY

In conclusion, it may be useful to summarize the main results presented in this Report.

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Equations (5) and (10) give general expressions for the rates of change of orbital inclination  i  and eccentricity  e  near resonance caused by a pertinent pair of harmonics, in terms of the resonance angle  $\phi$  defined in equation (3), and the functions  $\bar{F}$  and  G . Specific forms for the term of lowest degree  $\ell_0$ at the resonances most often encountered (15:1, 14:1, 16:1, 29:2 and 31:2) are given in section 4 for  di/dt  and in section 5 for  de/dt .  It is shown how these terms can be combined with those of higher degree ($\ell = \ell_0 + 2$, $\ell_0 + 4$, ...) into a lumped harmonic, defined in equations (16) and (17).

The function  $\bar{F}$ , which depends on inclination, is evaluated in section 6 by writing  $\bar{F} = AV$ , where explicit expressions for  A  are derived from the recurrence relation (25), and  V  is a normalizing constant given by (28).  Figs 1 and 2 give examples of the variation of  $\bar{F}$  with inclination.

The function  G , which depends on eccentricity, is most easily evaluated from a definite integral, equation (29), and a Fortran program (GQUAD) written by A.W. Odell for this purpose is listed in the Appendix. When  e  is small, a series expansion of  G  is useful, and explicit forms are given for  $\hat{G}$  (the first term in the series) based on equation (31).  Figs 3 and 4, which give the variation of  $G/\hat{G}$  with  e , show that  $\hat{G}$ , though useless as an approximation when  e  is large, is accurate to 1% if  e < 0.01 , as with most of the reson-ances that have been analysed.  The values of  G , rather than  $\hat{G}$ , have been used in a new determination[25] of harmonics of 15th order and 30th order: the main effect is that the 15th-order coefficients of odd degree are altered by about a quarter of a standard deviation, on average.

## Appendix

### A FORTRAN-77 PROGRAM, GQUAD, FOR COMPUTING G-FUNCTIONS BY QUADRATURE

#### A.1 Introductory remarks

As indicated in the main text, the functions of eccentricity, $G_{\ell pq}(e)$, emanate from the classical Hansen functions, $X_{\sigma}^{n,k}$; these are defined such that the true anomaly, $v$, can be related to the mean anomaly, $M$, by the expansion

$$\left(\frac{r}{a}\right)^n \exp(jkv) = \sum_{\sigma=-\infty}^{\infty} X_{\sigma}^{n,k}(e) \exp(j\sigma M) , \qquad (A-1)$$

where

$$\frac{r}{a} = \frac{1 - e^2}{1 + e \cos v} . \qquad (A-2)$$

Then the function $G_{\ell pq}$, as defined by Kaula[8], is just $X_{\sigma}^{n,k}$ with $n = -(\ell + 1)$, $k = \ell - 2p$ and $\sigma = k + q$. As noted in section 8, there would be advantages in writing $G_{\ell pq}$ as $G_{\ell q}^k$.

On multiplying both sides of equation (A-1) by $\exp\{-j(k + q)M\}$ and then integrating from 0 to $2\pi$, it follows that

$$G_{\ell pq} = \frac{1}{\pi} \int_0^{\pi} \left(\frac{a}{r}\right)^{\ell+1} \cos\{kv - (k + q)M\} \, dM . \qquad (A-3)$$

In equation (A-3) the true and mean anomalies, $v$ and $M$, are linked by the eccentric anomaly, $E$. Also, it is advantageous to integrate with respect to $E$, rather than $M$, if numerical quadrature is to be based on a uniform dissection of the independent variable. But

$$\frac{dM}{dE} = 1 - e \cos E = \frac{r}{a} , \qquad (A-4)$$

so we can rewrite (A-3) as

$$G_{\ell pq} = \frac{1}{\pi} \int_0^{\pi} \left(\frac{a}{r}\right)^{\ell} \cos\{kv - (k + q)M\} \, dE . \qquad (A-5)$$

Here $M$ is related to $E$ by Kepler's equation, the derivative of which gave (A-4), whilst $v$ is related to $E$ by the equation

$$\tan \tfrac{1}{2}v \;=\; \left(\frac{1 + e}{1 - e}\right)^{\tfrac{1}{2}} \tan \tfrac{1}{2}E \quad ; \qquad\qquad\qquad (A\text{-}6)$$

finally $a/r$ can be eliminated from (A-5) by using (A-4).

A.W. Odell has kindly provided a Fortran-77 program (in double precision) that implements the definite integral (quadrature) expressed in equation (A-5). This program, GQUAD, is listed in section A.2. The basis of the program is the uniform division of the interval $(0,\pi)$ into $N$ sub-intervals, over each of which the integral is computed from the four-strip Newton-Cotes formula,

$$\int_a^b f(x)\, dx \;\approx\; \frac{b - a}{90}\,(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) \quad , \qquad (A\text{-}7)$$

where $f_i = f\{a + \tfrac{1}{4}i(b - a)\}$ for $i = 0, 1, 2, 3, 4$. Equation (A-7) is exact for polynomials of degree up to 5, so the error in quadrature by GQUAD is $O(1/N^6)$. This can also be written $O(h^6)$, where $h$ is the width of each strip of a sub-interval (so that $4Nh = \pi$); but here we are not dealing with polynomials, and the convergence is much faster than this suggests.

The program requests the values of $e$, $k$, $\ell$ and $q$ as input, and then operates in one of two possible modes. The normal mode finds a suitable value of $N$ automatically, by starting with $N = 2$ and then successively doubling it. In this bisection process, successive estimates of the integral are compared until the value changes by less than 1 part in $10^6$. At this point the integral is deemed to have 'converged', and its final value, together with the final value of $N$, is printed; clearly, the value of $N$ can only be a power of 2.

The alternative mode of operation requires $N$ to be specified manually. The mode is selected by attaching a negative sign to $e$, whereupon the program requests that the value of $N$ be supplied. Any even integer is now legitimate for $N$, and there is only a single computation of the integral. In the computation, advantage is taken of a quasi-symmetry between $E$ and $\pi - E$, such that the quadrature routine appears to operate between $0$ and $\tfrac{1}{2}\pi$ rather than between $0$ and $\pi$. This is why $N$ must be even in the alternative mode of operation: it can then be halved to operate over the half-interval.

To provide examples of the manner in which $G_{\ell pq}$ diverges from $\hat{G}_{\ell pq}$, which is its value for $e = 0$, the ratio of the two is plotted in Fig 3 for various $(k,\ell,q)$ when $e$ ranges from 0 to 0.6. It is worth remarking that the GQUAD output can be used to divine explicit terms, beyond $\hat{G}_{\ell pq}$, in the series

in $e^2$ for $G_{\ell p q}$ . Thus, for two of the examples plotted, we have

$$G_{15,7,0} \approx 1 + 59e^2 + 0(e^4)$$

and

$$G_{17,8,0} \approx 1 + 75.5e^2 + 0(e^4) \ .$$

For the other two, the numerical coefficients are not clean-cut; but,
*approximately,*

$$G_{16,7,-1} \simeq 6.5e + 245e^3 + 0(e^5)$$

and

$$G_{14,7,1} \simeq 7.5e + 222e^3 + 0(e^5) \ .$$

Though the orbits analysed at resonance usually have $e < 0.01$ , GQUAD is valid
for large values of $e$ : for example, it gives $G_{15,7,0}(0.999) \approx 0.312042 \times 10^{43}$ ,
the value $N \approx 1024$ being required.

## A.2  Listing of Program GQUAD

```
C          PROGRAM GQUAD FOR ECCENTRICITY FUNCTIONS
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      COMMON /CGECC/ E, GI, GJ, GK
      EXTERNAL GECC
      PARAMETER (PI=3.1415926535897932D0, HPI=0.5D0*PI, EPS=1D-6)
    1 WRITE(1, *) 'GIVE: E, K, L, Q'
      READ(1, *) E, K, L, IQ
      IF(E.GE.1D0) STOP
      HMAX = PI/DMAX1(3D0, DABS(DFLOAT(K)), DABS(DFLOAT(IQ)))
      GI = L
      GJ = K
      GK = K + IQ
      IF (E.GT.0D0) THEN
        G = QINTC(GECC, 0D0, HPI, HMAX, EPS, N)/PI
       ELSE
        WRITE(1, *) 'GIVE: N (EVEN)'
        READ (1,*) N
        E = -E
        G = QINT(GECC, 0D0, HPI, N/2)/PI
C          (ONLY HALF N, BECAUSE QINT GOES ONLY TO HALF-PI)
      END IF
      WRITE(1, 2) G, N
    2 FORMAT ('G IS', G20.12, '  & N IS', I6)
      GO TO 1
      END


      DOUBLE PRECISION FUNCTION GECC(EE)
C          COMPUTES INTEGRAND FOR ECCENTRICITY FUNCTION
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      COMMON /CGECC/ E, GI, GJ, GK
      PARAMETER (PI = 3.1415926535897932D0)
      SQE = DSQRT((1D0 + E)/(1D0 - E))
      CEH = COS(EE*0.5D0)
      SEH = SIN(EE*0.5D0)
      CE = CEH*CEH - SEH*SEH
      SE = 2D0*SEH*CEH
      GECC = COS(2D0*GJ*DATAN2(SQE*SEH, CEH) - GK*(EE - E*SE))/
    1 (1D0 - E*CE)**GI
      GECC = COS(2D0*GJ*DATAN2(SQE*CEH, SEH) - GK*(PI - EE - E*SE))/
    1 (1D0 + E*CE)**GI + GECC
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION QINTC(F, A, B, H4MAX, EPS, N2)
C         INTEGRATES FUNCTION F FROM A TO B, BUILDING TO N ( = N2/2)
C         SUBINTERVALS BY BISECTION TO REFINE ACCURACY (EXACT FOR
C         QUINTICS). PROCESS STOPS WHEN STRIP LENGTH LESS THAN
C         H4MAX AND RELATIVE CHANGE IN INTEGRAL LESS THAN EPS.
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      H4 = B - A
      SC = H4/45D0
      S2 = (F(A) + F(B))*0.5D0
      S1 = F((A + B)*0.5D0)
      N2 = 2
    1 H2 = H4*0.5D0
      H = H2*0.5D0
      S = 0D0
      X = A + H
      DO 2 I = 1, N2
      S = F(X) + S
    2 X = X + H2
      S1 = S1 + S2
      QINTC = (16.*S + 6.*S1 + S2)*SC
C         TEST FOR CONVERGENCE
      IF (N2.GT.2 .AND. DABS(H4).LE.H4MAX .AND. DABS(QINTC - QINTCO)
     1 .LT.EPS*QINTCO) RETURN
      QINTCO = QINTC
      S2 = S1
      S1 = S
      N2 = N2 + N2
      H4 = H2
      SC = SC*0.5D0
      GO TO 1
      END


      DOUBLE PRECISION FUNCTION QINT(F, A, B, N)
C         INTEGRATES A FUNCTION FROM A TO B USING N STEPS
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      H = (B - A)/DFLOAT(N)
      H1 = 0.25D0*H
      H2 = H1*2D0
      H3 = H1*3D0
      C1 = H*7D0/90D0
      C2 = H*32D0/90D0
      C3 = H*12D0/90D0
      X = A
      F0 = F(X)
      QINT = 0D0
      DO 1 I = 1, N
      F4 = F(X + H)
      QINT = QINT + C1*(F0 + F4) + C2*(F(X + H1) + F(X + H3)) +
     1 C3*F(X + H2)
      F0 = F4
    1 X = A + DFLOAT(I)*H
      RETURN
      END
```

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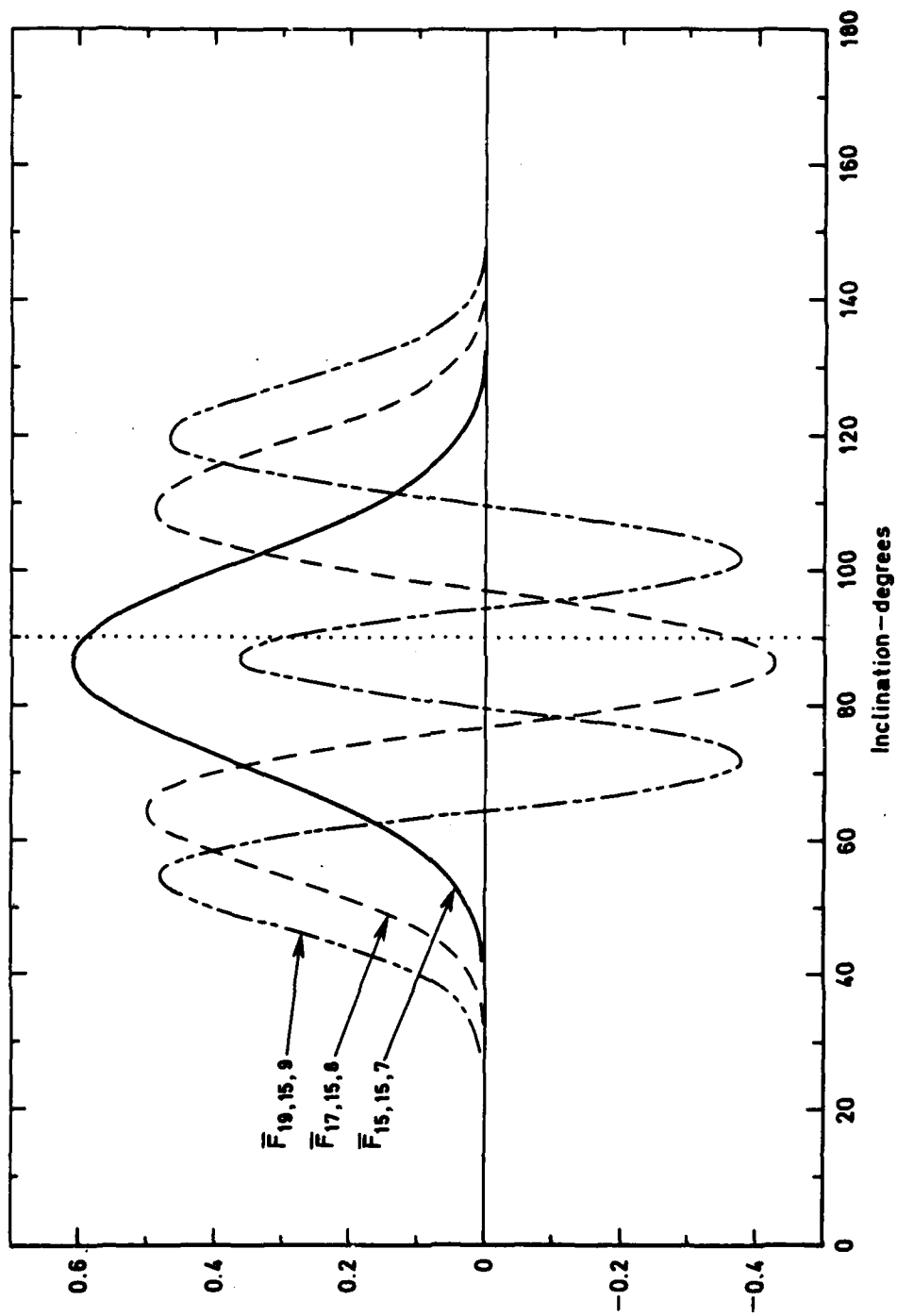
Fig 1

Fig 1

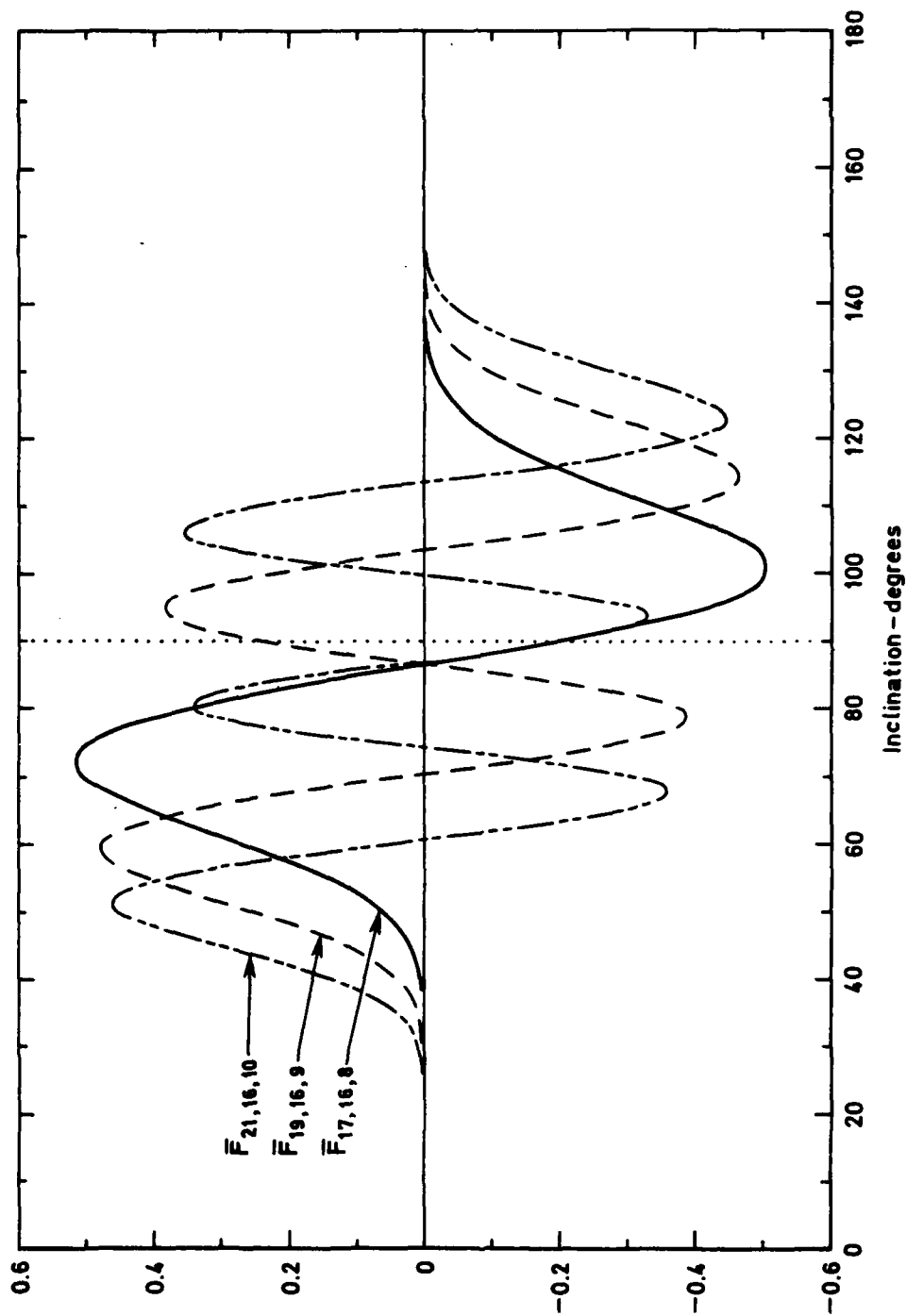Fig 1    Normalized inclination functions for 15:1 resonance

$\overline{F}_{19,15,9}$

$\overline{F}_{17,15,8}$

$\overline{F}_{15,15,7}$

Inclination — degrees

Fig 2



Fig 2    Normalized inclination functions for 16:1 resonance

$\overline{F}_{21,16,10}$
$\overline{F}_{19,16,9}$
$\overline{F}_{17,16,8}$

Inclination—degrees

Fig 3



Fig 3    Variation of $G_{\ell pq}/\hat{G}_{\ell pq}$ with $e$ for selected $(\ell, p, q)$, with $0 < e < 0.6$

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Fig 4



Fig 4    Variation of $G_{\ell pq}/\hat{G}_{\ell pq}$ with e for selected $(\ell, p, q)$, with $0 < e < 0.07$